JOURNAL OF REGIONAL SCIENCE, VOL. 22, NO. 3, 1982

A NOTE ON SUFFICIENT CONDITIONS FOR NEGATIVE EXPONENTIAL POPULATION DENSITIES

Jan K. Brueckner*

1. INTRODUCTION

The main testable prediction of the Mills-Muth model of urban spatial structure is that population density declines as distance to the urban center increases. The repeated confirmation of this prediction in empirical studies has led to a widespread consensus on the validity of the basic urban model, a rare achievement in applied economics. The most common specification of the population density function is negative exponential: density D is related to distance to the urban center x by the function $D = D_0 e^{-\gamma x}$, where $\gamma > 0$ and D_0 is population density at $x = 0$.[1] Although the negative exponential is a convenient functional form, most empirical investigators are aware that it is appropriate only under strong restrictions on the housing production technology, consumer tastes, and the nature of commuting costs. Moreover, it is common for researchers to cite Mills (1972) for the list of conditions which justify the function's use [see Kau and Lee (1976) and McDonald and Bowman (1976)]. Mills' conditions are (1) a Cobb-Douglas housing production function; (2) commuting costs which are linear in distance; and (3) a unitary price elasticity of housing demand. The purpose of this note is to point out that in spite of the pedagogic value of Mills' derivation, the last of these conditions is not strictly correct: the appropriate restriction on housing demand is that the *income compensated* price elasticity (not the regular price elasticity) is unitary. While this restriction is correctly stated by Muth (1969, Chap. 4), the failure of empirical researchers to grasp the difference between the elasticity restrictions (and the imprecision of Mills' version) is no doubt due to the lack of clarity of Muth's argument. In the next section of this note, analysis which is equivalent to Muth's is clearly presented, and the steps leading to Mills' restriction are discussed.[2]

2. ANALYSIS

The basic urban model assumes that the city is inhabited by individuals with uniform incomes and identical tastes over consumption of housing services q and a numeraire nonhousing good c. Commuting cost as a function of radial distance to

*Associate Professor of Economics, University of Illinois at Urbana-Champaign.

[1]For a recent study estimating such a function, see Glickman and Oguri (1978); for earlier studies, see Mills (1970) and Muth (1969).

[2]In private correspondence, Mills has indicated that he intentionally sacrificed precision in his textbook presentation in order to make the analysis more transparent to uninitiated readers.

Date received: August, 1981; revised, November, 1981.

the urban center is $t(x)$, so that disposable income at x is $y - t(x)$, where y is income at the center. Housing services are produced with capital N and land l according to the function $H(N, l)$, which is homogeneous of degree one (housing services are best viewed as the services derived from floor space, with H the production function for floor space). The rental price per unit of housing services (per square foot of floor space) is p, which depends on x, while the rental prices of capital and land are i and r respectively, with i constant over x but r free to vary.

Population density may be computed by noting that the ratio of square feet of housing per acre of land to square feet of housing per dwelling equals dwellings per acre of land. Thus the ratio $(H/l)/q$ equals dwellings per acre of land and is proportional to population density when households are of uniform size. Now the appropriateness of the negative exponential density function depends on the form of the x-derivative of the natural logarithm of density. If this derivative is constant over x, then densities will follow the negative exponential function. Letting * denote natural logarithm, the derivative of interest is

$$\frac{1}{D}\frac{dD}{dx} \equiv \frac{dD^*}{dx} = \frac{d[(H/l)/q]^*}{dx} = \frac{d(H/l)^*}{dx} - \frac{dq^*}{dx} \tag{1}$$

The following analysis will indicate when (1) will indeed be constant over x. The first step is to calculate $d(H/l)^*/dx$ making use of the housing producer's first-order conditions. Housing producer profit is $pH(N, l) - iN - rl$, and the necessary conditions for a maximum are $pH_1 = i$ and $pH_2 = r$ (subscripts denote partial derivatives). These equations together imply $H_2(N, l)/H_1(N, l) = r/i$, and the zero-degree homogeneity of H_1 and H_2 allows this equation to be rewritten as $H_2(N/l, 1)/H_1(N/l, 1) = r/i$. The last equation determines the capital-land ratio N/l solely as a function of the factor price ratio r/i (this, of course, is a reflection of the fact that constant returns functions are homothetic). Now the dependence of H/l on x follows from the dependence of N/l on r/i and the dependence of r on x. To see this, note first that

$$\begin{aligned} \frac{d(H/l)^*}{dx} &= \frac{l}{H}\frac{dH(N/l, 1)}{dx} \\ &= \frac{l}{H}H_1(N/l, 1)\frac{d(N/l)}{dx} \\ &= \frac{NH_1(N/l, 1)}{H}\frac{d(N/l)^*}{dx} \end{aligned} \tag{2}$$

where the first equality uses $H(N, l)/l = H(N/l, 1)$. Now since $H_1 = i/p$ and N/l is a function only of r/i, (2) may be written

$$\frac{iN}{pH}\frac{d(N/l)^*}{d(r/i)^*}\frac{d(r/i)^*}{dx} = \mu_N \sigma_{Nl}\frac{dr^*}{dx} \tag{3}$$

where $\mu_N \equiv iN/pH$ is capital's factor share in housing production and σ_{Nl} is the elasticity of substitution between capital and land in housing (note that $di^*/dx \equiv 0$ and $d(N/l)^*/d(r/i)^* = [d(N/l)^*/d(H_2/H_1)^*][d(H_2/H_1)^*/d(r/i)^*] = \sigma_{Nl} \cdot 1$).

While (3) shows how $d(H/l)^*/dx$ is related to dr^*/dx, further analysis establishes the connection between $d(H/l)^*/dx$ and dp^*/dx. Total differentiation of the zero-profit identity $pH - iN - rl \equiv 0$ (profits are identically zero by constant returns) yields

$$\frac{dp}{dx}H - \frac{dr}{dx}l + (pH_1 - i)\frac{dN}{dx} + (pH_2 - r)\frac{dl}{dx} = \frac{dp}{dx}H - \frac{dr}{dx}l = 0 \tag{4}$$

using the producer's first-order conditions. Rearrangement of (4) yields

$$\frac{dr^*}{dx} = \frac{H}{rl}\frac{dp}{dx} = \frac{1}{\mu_l}\frac{dp^*}{dx} \tag{5}$$

where $\mu_l \equiv rl/pH$ is land's share in housing production. Combining (3) and (5) then gives

$$\frac{d(H/l)^*}{dx} = \frac{\mu_N \sigma_{Nl}}{\mu_l}\frac{dp^*}{dx} \tag{6}$$

Computation of dp^*/dx makes use of the first-order conditions for the consumer optimization problem. The problem is to maximize the utility function $u(c, q)$ subject to $c + pq = y - t(x)$ by choice of c, q, and x. Substitution yields the maximand $u(y - t - pq, q)$ and the first-order conditions $u_2/u_1 = p$ and

$$\frac{dt}{dx} + q\frac{dp}{dx} = 0 \tag{7}$$

Rearrangement of (7) then yields

$$\frac{dp^*}{dx} = \frac{-dt/dx}{m} < 0 \tag{8}$$

where $m \equiv pq$ is expenditure on housing.

Now in order for the identical urban residents to live voluntarily at different locations within the city, each individual must be locationally indifferent. That is, every location must be optimal, implying that condition (7) must hold at all urban values of x. Locational indifference will not obtain, of course, unless utilities are uniform across x. It is easy to see, however, that joint satisfaction of (7) and the other first-order condition at all x guarantees x-invariant utilities. This follows from totally differentiating $u(y - t - pq, q)$, which yields $-u_1(dt/dx + qdp/dx) + (u_2 - pu_1)dq/dx$, a quantity which equals zero at all x when (7) and $u_2/u_1 = p$ hold everywhere. Thus, the spatial variation in p implied by (8) reconciles urban residents to differences in commuting costs; the decline of p with x cancels the utility-decreasing effect of longer commutes and leaves consumers locationally indifferent.

The final step in deriving dD^*/dx is the computation of dq^*/dx, a task which is immediate in light of the preceding discussion. Since the decline with x in the price per square foot of housing keeps consumer utilities constant in the face of higher commuting costs, it is clear that the change in q caused by x-induced changes in p and disposable income follows from movement along an *income-compensated* (constant utility) demand curve. Therefore

$$\frac{dq^*}{dx} \equiv \frac{1}{q}\frac{dq}{dx} = \frac{1}{q}\frac{\partial q}{\partial p}\bigg|_{u=\text{constant}} \cdot \frac{dp}{dx} = \eta \frac{dp^*}{dx} \tag{9}$$

where η is the income-compensated elasticity of demand. Substituting (6) and (9) into (1) then yields

$$\begin{aligned} \frac{dD^*}{dx} &= \left(\frac{\mu_N \sigma_{Nl}}{\mu_l} - \eta\right)\frac{dp^*}{dx} \\ &= -\left(\frac{\mu_N \sigma_{Nl}}{\mu_l} - \eta\right)\frac{dt/dx}{m} \end{aligned} \tag{10}$$

where the second equality uses (8). Since all terms in (10) are positive except for η, (10) is unambiguously negative; population density declines with x. The intuition for this result is easily stated: lower housing prices at greater distances lead to larger dwellings ($dq^*/dx = \eta dp^*/dx > 0$) while lower land rents at greater distances make lower capital-land ratios optimal, and thus lead to fewer square feet of housing per acre ($d(H/l)^* = \mu_n \sigma_{Nl} dr^*/dx < 0$). Together, these effects yield fewer dwellings per acre and lower population densities at greater distances.

It is now possible to consider the central question of this note: when will (10) equal a constant, leading to a negative exponential density function? A sufficient condition for this outcome is clearly that each element in (10) is itself independent of x.[3] In other words, μ_N, μ_l, σ_{Nl}, η, dt/dx, and m all should be invariant with x (note that the factor shares and elasticities will *not* in general be independent of x). Now if the housing production function is Cobb-Douglas ($H(N, l) \equiv N^{\alpha} l^{1-\alpha}$), then $\sigma_{Nl} = 1$, $\mu_N = \alpha$, and $\mu_l = 1 - \alpha$. Moreover, if $t(x) \equiv \phi + \delta x$, then $dt/dx = \delta$. If in addition the income compensated demand elasticity η is constant and if movement along the income-compensated housing demand curve yields constant expenditures m, then the sufficient conditions for constancy of (10) will be met. Clearly, $\eta = -1$ leads to satisfaction of both these requirements since a unitary price elasticity yields expenditures which are independent of price and thus constant over x. In summary, sufficient conditions for the validity of the negative exponential density function are (1) a Cobb-Douglas housing production function; (2) commuting costs which are linear in distance; and (3) a unitary income-compensated price elasticity of housing demand.

Although Mills (1972) imposes specific functional forms at the beginning rather than at the end of his derivation, much of his analysis parallels the above. He implicitly calculates $d(H/l)^*/dx$ as in (6), and condition (7) enters importantly in his argument. Mills' derivation differs from the above, however, in the computa-

[3]Although this is obviously not a necessary condition for the constancy of (10), it is very hard to imagine a situation in which the elements of (10) are nonconstant over x while (10) itself is x-invariant.

tion of dq^*/dx. First, he assumes that the demand function for housing has the constant elasticity form $q = AI^\theta p^\beta$, where I is income and p as before is price. While such a demand function is perfectly acceptable, Mills' illegitimate step is to substitute *gross income* y instead of *disposable income* $y - t(x)$ in place of I. The resulting demand relationship $q = Ay^\theta p^\beta$ ignores the crucial fact that consumer purchasing power declines with x. Calculation of dq^*/dx based on this inappropriate relationship yields $dq^*/dx = (\partial q^*/\partial p^*)(dp^*/dx) = \beta dp^*/dx$. Repeating the previous substitutions then gives

$$\frac{dD^*}{dx} = -\left(\frac{\mu_N \sigma_{Nl}}{\mu_l} - \beta\right)\frac{dt/dx}{m} \tag{11}$$

Note in (11) that the uncompensated price elasticity β incorrectly takes the place of the compensated elasticity η [compare (10) and (11)]. Now since the above demand relationship implies $m = Ay^\theta p^{1+\beta}$, a unitary price elasticity ($\beta = -1$) implies $m = Ay^\theta$. Imposing in addition the assumptions of a Cobb-Douglas production function [so that $\mu_N\sigma_{Nl}/\mu_l = \alpha/(1 - \alpha)$] and linear commuting costs (implying $dt/dx = \delta$), substitution in (11) yields $dD^*/dx = -[\alpha/(1 - \alpha) + 1]\delta/Ay^\theta = -\delta/(1 - \alpha)Ay^\theta$, a constant.

This discussion shows how misspecification of the demand relationship by use of an improper income variable leads to the erroneous conclusion that a unitary uncompensated price elasticity, together with the other restrictions, implies a constant dD^*/dx. To show explicitly that this elasticity assumption will not yield the desired result when the correct income variable is used, note that when $q = A[y - t(x)]^\theta p^\beta$, it follows that

$$\begin{aligned}\frac{dq^*}{dx} &= -\frac{\theta}{y - t}\frac{dt}{dx} + \beta\frac{dp^*}{dx}\\ &= (\theta\epsilon_q + \beta)\frac{dp^*}{dx}\end{aligned} \tag{12}$$

where $\epsilon_q \equiv m/(y - t)$ is the expenditure share for housing (the second inequality uses $dt/dx = -mdp^*/dx$). Imposing a Cobb-Douglas production function and linear commuting costs and using (12) with $\beta = -1$, familiar steps then give

$$\begin{aligned}\frac{dD^*}{dx} &= -\left[\frac{\alpha}{1 - \alpha} - (\theta\epsilon_q - 1)\right]\frac{\delta}{m}\\ &= -\left\{\frac{1}{1 - \alpha} - \theta A[y - t(x)]^{\theta - 1}\right\}\frac{\delta}{A[y - t(x)]^\theta}\end{aligned} \tag{13}$$

a quantity which is clearly not constant over x [note $m = A(y - t)^\theta$ and $\epsilon_q = A(y - t)^{\theta - 1}$ when $\beta = -1$]. The nonconstancy of (13) is, of course, simply a reflection of the fact that a unitary *uncompensated* price elasticity generally implies a nonunitary *compensated* elasticity, which from above must yield a nonconstant dD^*/dx. Indeed, it is easy to show that $\theta\epsilon_q - 1$, which appears in the position of η in (10), is precisely the compensated price elasticity for the given demand function when $\beta = -1$.[4] The obvious restriction $\theta > 0$ (housing is a normal good) implies that the compensated elasticity $\theta\epsilon_q - 1$ exceeds -1, accounting for the nonconstancy of dD^*/dx in (13).

A wealth of empirical evidence on the parameters of housing demand is now in existence [see Mayo (1981) for a recent survey]. An important question is whether this evidence justifies use of the negative exponential density function in empirical studies (in other words, is the evidence consistent with $\eta = -1$?). Unfortunately, the answer to this question is unfavorable: nearly all studies show that the uncompensated price elasticity of housing demand exceeds -1, implying that the compensated elasticity is substantially greater than -1. In fact, averaging the joint income and price elasticity estimates for owner-occupiers summarized in Mayo's Table 1A and assuming a housing expenditure share of 0.15, the implied value of η is -0.58. The magnitude of this number suggests that any negative exponential population density regression will involve specification error.[5]

As a final observation, it is interesting to note that the preceding discussion necessitates reinterpretation of the results of one important empirical study of population densities. Recognizing the possibility of specification error, Kau and Lee (1976) used an ingenious application of the Box-Cox estimation procedure to test for the appropriateness of the negative exponential function rather than imposing it a priori. However, since they invoked Mills' assumptions, Kau and Lee interpreted a divergence from the negative exponential form as evidence of a nonunitary uncompensated price elasticity of housing demand. Under this interpretation, their empirical results implied $\beta > -1$ for half the sample cities, with $\beta = -1$ holding for the remainder, conclusions which appear consistent with the independent evidence of price inelasticity. As should be clear from the above discussion, however, the correct inference to be drawn from Kau and Lee's results concerns the compensated price elasticity of housing demand. Correctly interpreted, their results show that this elasticity frequently exceeds -1, a conclusion which need not imply overall price inelasticity.

REFERENCES

Brueckner, Jan. K. "A Vintage Model of Urban Growth," *Journal of Urban Economics*, 8 (1980), 389–402.

Glickman, Norman J. and Yukio Oguri. "Modeling the Urban Land Market: The Case of Japan," *Journal of Urban Economics*, 6 (1978), 505–525.

---

[4]Using the Slutsky equation

$$\frac{\partial q}{\partial p} = \left.\frac{\partial q}{\partial p}\right|_{u\,=\,\text{constant}} - q\frac{\partial q}{\partial I}$$

it follows that

$$\eta = \epsilon_q \frac{I}{q}\frac{\partial q}{\partial I} + \frac{p}{q}\frac{\partial q}{\partial p} = \theta\epsilon_q - 1$$

[5]There are, of course, other factors which can cause densities to diverge from the negative exponential form. Most important among these are housing vintage effects and the presence of multiple income groups [see Brueckner (1980) and Johnson and Kau (1980)].

Johnson, S. R. and James B. Kau. "Urban Spatial Structure: An Analysis with a Varying Coefficient Model," *Journal of Urban Economics*, 7 (1980), 141–154.
Kau, James B. and Cheng F. Lee. "Functional Form, Density Gradient and Price Elasticity of Demand for Housing," *Urban Studies*, 13 (1976), 193–200.
Mayo, Stephen K. "Theory and Estimation in the Economics of Housing Demand," *Journal of Urban Economics*, 10 (1981), 95–116.
McDonald, John F. and H. Woods Bowman. "Some Tests of Alternative Urban Population Density Functions," *Journal of Urban Economics*, 3 (1976), 242–252.
Mills, Edwin S. "Urban Density Functions," *Urban Studies*, 7 (1970), 5–20.
———. *Urban Economics*. Glenview, Illinois: Scott Foresman, 1972.
Muth, Richard F. *Cities and Housing*. Chicago: University of Chicago Press, 1969.